Biomedical Signal Processing

- Wavelet Analysis: Time Domain Properties (1) -

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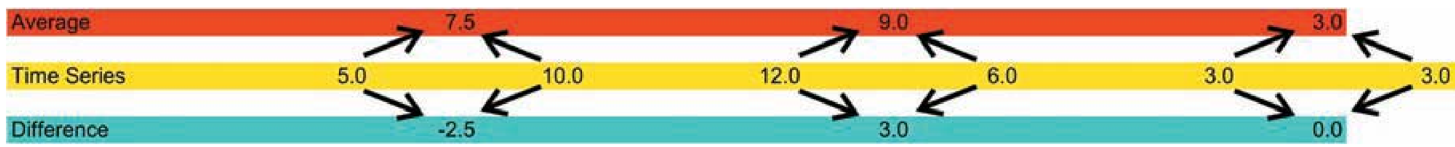
Introduction

- History and applications of wavelet analysis
  - The mathematics for wavelet analysis – existed for about a century
  - Most of applications in signal processing, feature detection, and data compression – developed over the past few decades
  - Analyzing physiological systems
    - Wavelet analysis – very useful
    - It provides the means to detect and analyze *non-stationarity in signals* (as opposed to most classical signal analysis approaches)

- Types of wavelets
  - The simplest: Haar wavelet (first described in the early 1900s by Alfred Haar)
  - A few other famous, more recent contributors: Morlet, Mallat, and Daubechies, …

Wavelet Transform

- A sampled time series: 5.0, 10.0, 12.0, 6.0, 3.0, 3.0, …
- Examined for trends and fluctuations over subsequent pairs



  - Average (red) of subsequent pairs = [x(n-1)+x(n)]/2
  - Difference (blue) = [x(n-1) - x(n)]/2
- No lost information (∵ the original time series (yellow) can be reconstructed from a combination of the average and difference vectors)
  - First value = the sum of average and difference (5 = 7.5 – 2.5)
  - Second value = the difference (10 = 7.5 – (-2.5))

  …

Wavelet Transform

- Time series – the same information as the average and difference signals
  - Represented either in original raw form [5.0, 10.0, 12.0, 6.0, 3.0, 3.0, …] or
  - in a transformed fashion as a combination of the average and difference form [7.5, 9.0, 3.0, …][-2.5, 3.0, 0.0, …]

- Application of the Haar wavelet – almost identical to calculating the average and difference as above procedure (aside from a factor of $\sqrt{2}$)

Wavelet Transform

- Haar Wavelet and Scaling Signals
  - Level-1 Haar wavelet & associated scaling signal (Fig. 15.1)
    - Square waves with amplitudes of $1/\sqrt{2}$
    - Wavelet – biphasic & scaling signal – non-negative
  - The transform of an input signal of N samples
    - First step:
      - To define the level-1 Haar wavelet (W) and scaling signal (S) as vectors of length N
      - $W_1^1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, \ldots, 0\right], \quad S_1^1 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \ldots, 0\right]$
        - Superscripts → indicating level-1
        - Subscripts → indicating the position of both signals in the vectors

Wavelet Transform

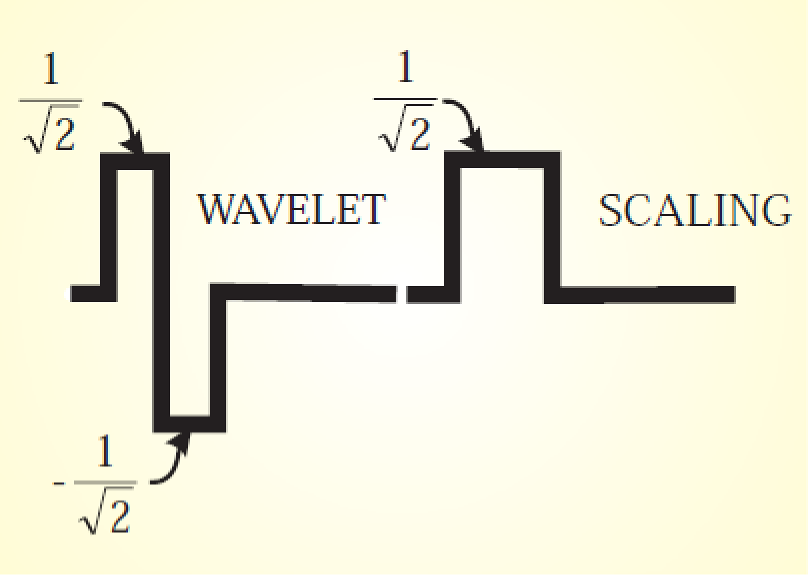


Figure 15.1 The level-1 Haar wavelet and scaling signal.

Wavelet Transform

- Haar Wavelet and Scaling Signals
  - The transform of an input signal of N samples
    - Scaling → The trend (= weighted average), wavelet → the fluctuation (= weighted difference) in a time series
      - Function G – sampled N times at regular time interval:

$$G = [g_1, g_2, g_3, \ldots, g_N]$$

      - Trend of the first 2 points:

$$t_1 = \frac{g_1 + g_2}{\sqrt{2}} = G.S_1^1$$ (scalar product of G and S_1^1)

      - Fluctuation btw the first 2 points:

$$f_1 = \frac{g_1 - g_2}{\sqrt{2}} = G.W_1^1$$ (scalar product of G and W_1^1)

Wavelet Transform

- Haar Wavelet and Scaling Signals

  < Notes >

  - Reason for the weighting factor
    - Division by $\sqrt{2}$ instead of simply dividing by 2
    - To preserve the energy content across the transformed variables
  - <u>Trend</u>
    - The sum of g_1 and g_2 weighted by $1/\sqrt{2}$
    - Meaning: the <u>average</u> of the 2 data points – multiplied by $\sqrt{2}$

$$\left(\text{i.e., } \frac{g_1+g_2}{2}\sqrt{2} = \frac{g_1+g_2}{\sqrt{2}}\right)$$

  - The same relationship btw the <u>fluctuation</u> and the <u>difference</u>

Wavelet Transform

- Haar Wavelet and Scaling Signals
  - Continuing with the procedure
    - Shifting wavelet and scaling signals by two positions:
      - Wavelet: $W_2^1 = \left[0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \ldots, 0\right]$
      - Scaling: $S_2^1 = \left[0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \ldots, 0\right]$
      - Subscripts for S and W: 2 → reflecting the shift of the signals to the second pair of data points
    - Repeating the process of calculating the trend and fluctuation
      - Weighted average of points 3 and 4: $t_2 = G.S_2^1$
      - Weighted difference btw points 3 and 4: $f_2 = G.W_2^1$

Wavelet Transform

- Haar Wavelet and Scaling Signals
  - Continuing to shift the wavelet and scaling signals in steps of 2 until the end of signal
    - Length of G = N → N/2 trend values & N/2 fluctuation values
    - Expression for the trend values:

$$t_m = \frac{g_{2m-1} + g_{2m}}{\sqrt{2}} = G.S_m^1 \qquad (m = 1, 2, 3, \ldots, N/2)$$

    - Weighted difference btw subsequent pairs of points (fluctuation values):

$$f_m = \frac{g_{2m-1} - g_{2m}}{\sqrt{2}} = G.W_m^1 \qquad (m = 1, 2, 3, \ldots, N/2)$$

Wavelet Transform

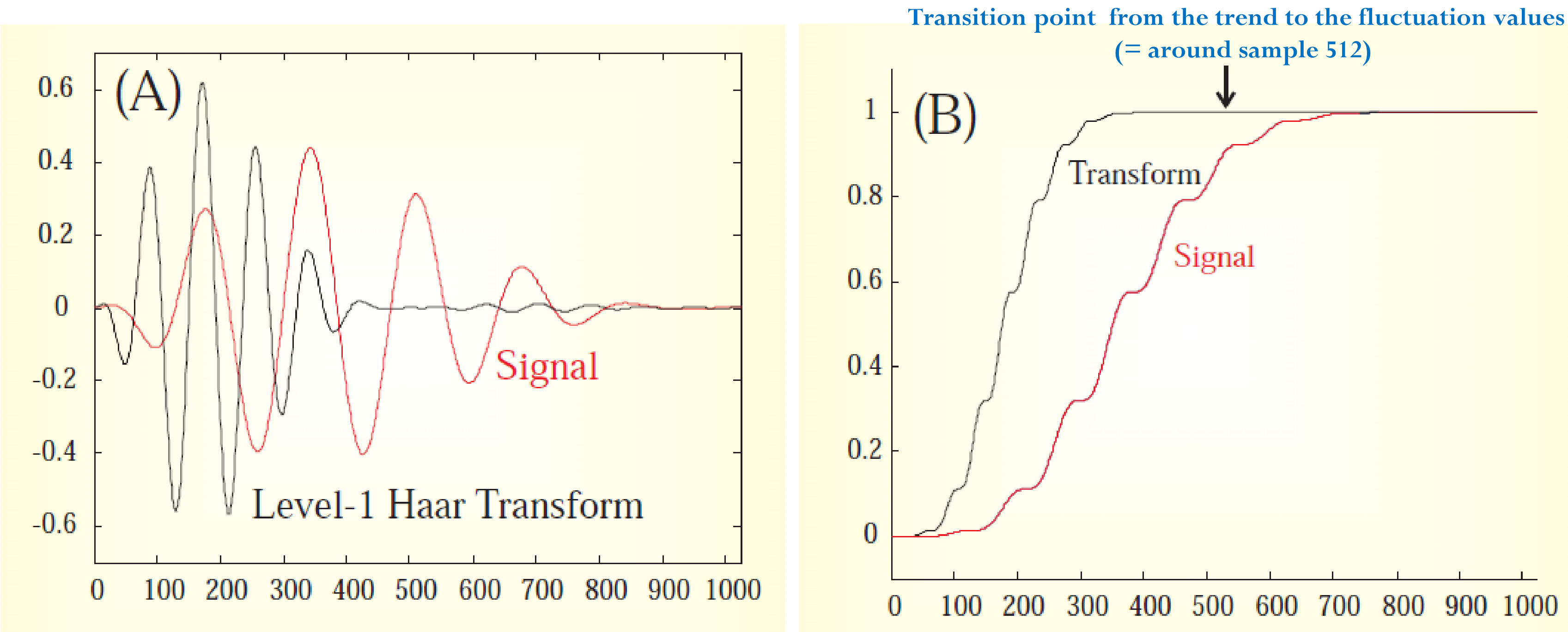
- Haar Wavelet and Scaling Signals
  - Grouping all the weighted averages and differences into two vectors
    - $a^1 = [t_1, t_2, \ldots, t_{N/2}]$
    - $d^1 = [f_1, f_2, \ldots, f_{N/2}]$
    - Superscript of a and d → indicating level-1 vectors
    - Subscript of the elements t and f → indicating the position of the wavelet and scale signals within the original N data points

Wavelet Transform

- Level-1 Haar Transform and the Inverse Haar Transform
  - Level-1 Haar transform
    - Defined from the preceding procedure to determine the trend and fluctuation components of a signal
    - The level-1 Haar transform of G = a^1 and d^1 :

$$G \overset{H_1}{\longmapsto} (a^1 \mid d^1)$$

    - Example time series [5.0, 10.0, 12.0, 6.0, 3.0, 3.0]
      - Level-1 Haar transform → two vectors:
        - $[7.5\sqrt{2}, 9.0\sqrt{2}, 3.0\sqrt{2}]$
        - $[-2.5\sqrt{2}, 3.0\sqrt{2}, 0.0]$
      - A result very similar to the averages and differences



(trend: 1 ~ 512, fluctuation: 513 ~ 1024)

Figure 15.2 Example of a level-1 Haar transform. (A) The transform (black) of an input wave (red). (B) The cumulative energy shows that for the transformed wave the trend signal contains most of the energy — that is, at point 512 (arrow) the ratio is ~1 (100%). (very little energy shows up in the fluctuation signal, e.g., $f_m \approx 0$)

Wavelet Transform

- Level-1 Haar Transform and the Inverse Haar Transform
  - Level-1 Haar transform
    - Transform result (Fig. 15.2)
      - First half: the trend (1~512)
        - Produced with the scaling signal S
        - Containing high amplitudes
      - Second part: the fluctuation (513~1024)
        - Produced with the wavelet W
        - Producing a low amplitude signal

      - Seeming a trivial observation, but a critical aspect of the analysis

Wavelet Transform

- Level-1 Haar Transform and the Inverse Haar Transform
  - Inverse Haar transform
    - Starts from the a^1 and d^1 transformed vectors
    - Allowing to create the original function G again
    - The inverse procedure – expressed in the form of a summation with the definition of A^1 and D^1 as:
      - $A^1 = [t_1, t_1, t_2, t_2, \ldots\ldots\ldots\ldots, t_{N/2}, t_{N/2}]/\sqrt{2}$
      - $D^1 = [f_1, -f_1, f_2, -f_2, \ldots\ldots\ldots\ldots\ldots, f_{N/2}, -f_{N/2}]/\sqrt{2}$

      - Doubling of t_n and f_n – not typos
      - Each second f_n of the pair – associated with a (-) sign
      - All the terms – divided by $\sqrt{2}$ (correct for the fact that the average and difference were multiplied by $\sqrt{2}$)

Wavelet Transform

- Level-1 Haar Transform and the Inverse Haar Transform
  - Inverse Haar transform
    - The inverse Haar transform = the sum of both vectors: $\boxed{G = A^1 + D^1}$
    - (more formal) vector form of A^1 and D^1:
      - $A^1 = t_1 S_1^1 + t_2 S_2^1 + \ldots + t_{N/2} S_{N/2}^1$

        $= (G.S_1^1)S_1^1 + (G.S_2^1)S_2^1 + \ldots + (G.S_{N/2}^1)S_{N/2}^1$
        - $(G.S_m^1)$: a scalar product representing t_m
        - $(G.S_m^1) \times S_m^1$ → a vector $[0, 0, \ldots, t_m, t_m, \ldots, 0]/\sqrt{2}$
          - The sum of these vectors for all m → express for A^1 in the previous page !
      - $D^1 = f_1 W_1^1 + f_2 W_2^1 + \ldots + f_{N/2} W_{N/2}^1$

        $= (G.W_1^1)W_1^1 + (G.W_2^1)W_2^1 + \ldots + (G.W_{N/2}^1)W_{N/2}^1$
        - The same procedure – for the wavelet to obtain this equation
      - Advantage of this notation
        - Easily extended from level 1 to a higher level transform
        - Also applied to inversion of other transforms besides Haar transform

Wavelet Transform

- Energy of the Level-1 Transform
  - Haar transform
    - Fairly simple ($\because$ a weighted average and weighted difference)
    - Only apparent nuisance: the $\sqrt{2}$ factor appearing in the wavelet definition, the transform, and the inverse transform
  - The reason for $\sqrt{2}$ correction for the level-1 Haar transform
    - The conservation of energy across domain
    - To keep the energy content of the signal the same across the signal transformations and inverse transformations as with the Fourier transform (Parseval's theorem)
  - Higher levels of the Haar transform → different normalization factor other than $\sqrt{2}$

Wavelet Transform

- Energy of the Level-1 Transform
  - Definition of the energy of a sample = the square of the sampled value
    - Energy preservation in the level-1 transform: $g_1^2 + g_2^2 = t_1^2 + f_1^2$
      - $g_1^2 + g_2^2$: the energy of the first two samples of G
      - t_1 and f_1 : the first elements derived in the transform from these samples
    - For all pairs and their associated trend and fluctuation values
      - $$t_m = \frac{g_{2m-1} + g_{2m}}{\sqrt{2}} \Rightarrow t_m^2 = \frac{g_{2m-1}^2 + g_{2m}^2 + 2g_{2m-1}g_{2m}}{2}$$
      - $$f_m = \frac{g_{2m-1} - g_{2m}}{\sqrt{2}} \Rightarrow f_m^2 = \frac{g_{2m-1}^2 + g_{2m}^2 - 2g_{2m-1}g_{2m}}{2}$$

      $\rightarrow$ $t_m^2 + f_m^2 = g_{2m-1}^2 + g_{2m}^2$ (for all m)
  - Inverse WT procedure
    - Exactly recreates the original function G
    - The transform and its inverse – also preserve the energy content!