Biomedical Signal Processing

- Wavelet Analysis: Time Domain Properties (2) -

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Prof: Boreom Lee

Wavelet Transform

- Multiresolution Analysis (MRA)
  - Higher-level transforms (Fig. 15.3)
    - Procedure for the level-1 Haar transform – can be repeated multiple times
    - Obtained by recursively applying the transform to the trend signal (the weighted average)
      - We leave the fluctuation (weighted difference) intact
      - Continue to split the weighted average only!

< Note >

- MRA – only one possible approach
- Wavelet packet transform
  - Transforms both the trends and fluctuations
  - Wavelet packet transform using the Haar scaling and wavelets → the Walsh transform

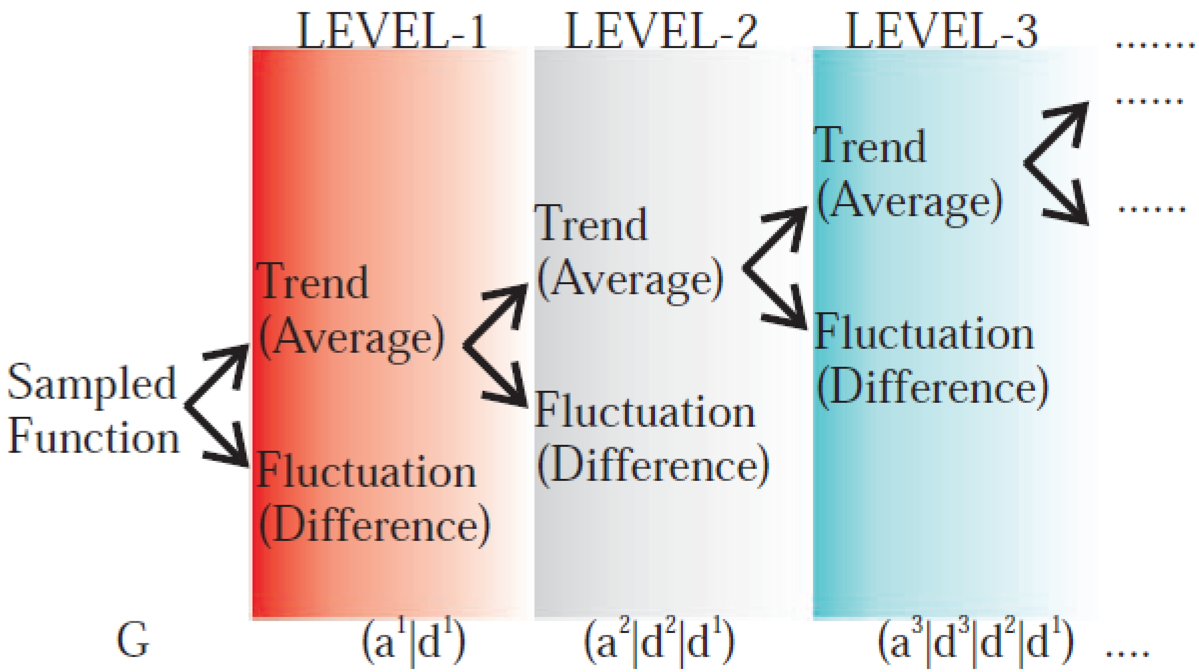

Figure 15.3 Higher-level Haar transforms of a sampled function *G*.

Wavelet Transform

- Multiresolution Analysis (MRA)
  - Using the transform $G \overset{H_1}{\mapsto} (a^1 | d^1)$ repeatedly:

$$\begin{array}{lll} G \overset{H_1}{\longmapsto} & (a_1 & d_1) \\ & \;\;\lfloor\!\overset{H_1}{\longrightarrow} & (a_2 \quad d_2) \end{array}$$

    - The level-2 transform: $G \overset{H_2}{\mapsto} (a_2\ d_2\ d_1)$
      - Combining results after using the level-1 transform twice
    - The level-n transform: $G \overset{H_n}{\mapsto} (a_n\ d_n\ d_{n-1} \ldots d_1)$
  - Multiresolution analysis (MRA)
    - Simply using the algorithm from the level-1 Haar program multiple times → repeated action !

The same input signal as the one for the level-1 Haar transform

The first trend signal which corresponds with the left half of the level-1 Haar transformed signal

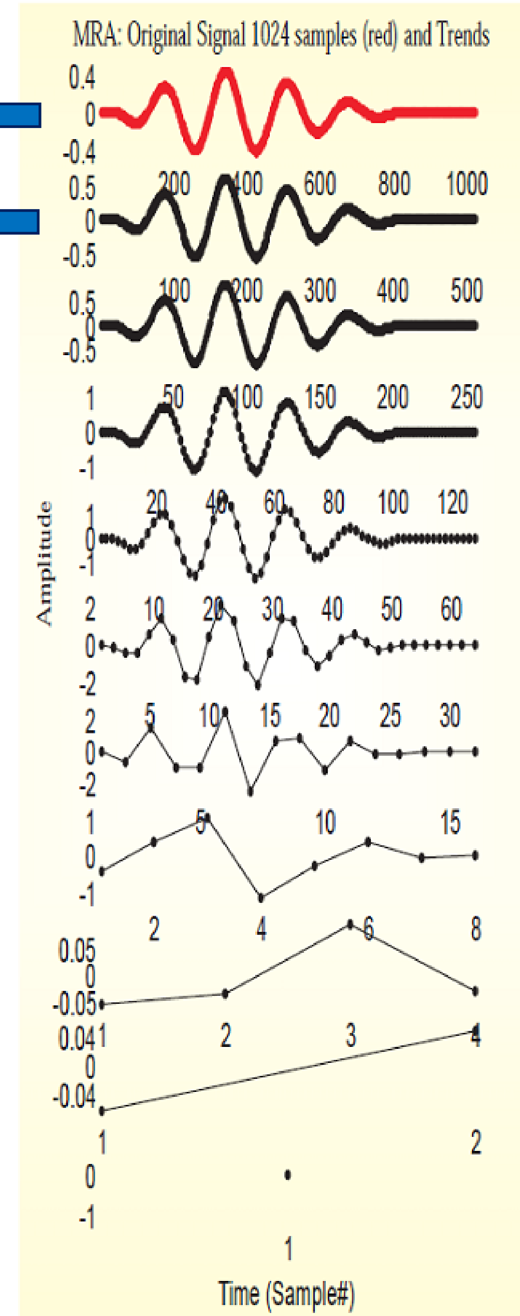

Figure 15.4 Multiresolution analysis (MRA) showing the trends (averages) of subsequent levels of the Haar transform using the procedure depicted in Figure 15.3. The fluctuation signals are not shown here but can be obtained from MATLAB script pr15_2.m.

Wavelet Transform

- Multiresolution Analysis (MRA)
  - The idea of MRA: to repeatedly use the level-1 transforms → effectively leading to higher-level scaling and wavelet signals
  - Direct formulation of higher-level transforms
    - For the sake of computational efficiency
    - Start with <u>a general form of the wavelet</u>
      - Haar → biphasic square wave in continuous time:

$$W_H(t) = \begin{cases} 1 & if \quad 0 \le t < \frac{1}{2} \\ -1 & if \quad \frac{1}{2} \le t < 1 \\ 0 & otherwise \end{cases}$$

        - W_H : called mother wavelet
    - WT of an input time series
      - By **translating** the wavelet operation over the input
      - By using wavelets of different scales → <u>different levels of the WT</u>
        - To study the correlation at different scales: mother wavelet – stretched (**dilated**)

Wavelet Transform

- Multiresolution Analysis (MRA)
  - The dilation *k* and translation *n* of the mother wavelet:

$$W(t)_n^k = \frac{1}{\sqrt{2^k}} W_H\left(\frac{t - 2^k n}{2^k}\right)$$

    - Discrete time version
      - The support (set of time indices where the wavelet is nonzero) for the k-level wavelet = 2^k
      - Example: the level-2 Haar wavelet
        - $$W_1^2 = \left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 0, \ldots, 0\right]$$
        - Comparing with the level-1 Haar wavelet W_1^1
          - The nonzero values – change by a factor of $1/\sqrt{2^k}$
          - The support – increase2 from 2 to 4

Wavelet Transform

- Application of Wavelets in Signal Compression and Denoising

  1. Data compression (Figure 15.4)
     - Features of the original waveform – preserved in fewer samples → the energy compaction property of the Haar transform – usable for data compression purposes
     - Original signal – 1024 samples → each subsequent trend signal – reduces the number of samples by a factor of 2
     - Compression at some point – accomplished at the cost of detail in the signal (energy stored in the fluctuation parts of the transform)
       - First steps reducing 1024 to 256 points – preserve the overall signal features rather well
       - Further compression beyond 256 points → severe loss of signal properties

  2. Removal of a noise
     - Example: a signal contaminated with high-frequency noise
     - The fluctuation part of the transform – mainly represent the noise component
     - Removal of the fluctuation signal followed by an inverse transform → an efficient approach to "clean up" time series and pictures

Other Wavelet Functions

- A large set of different wavelets and wavelet analysis packages – available in signal processing
- Depending on their purpose (signal compression, detection of transient phenomena in the time domain, quantifying instantaneous frequency components, etc)
  - Real form: Haar wavelet
  - Complex form: Morlet wavelet
  - Even symmetric form: Mexican Hat wavelet
  - Odd symmetric form: Haar wavelet
- Rich set of types of varied wavelet & associated scaling signals – possible ($\because$ they only have to satisfy a few fairly reasonable conditions → see next slide)
- **Daubechies wavelet** (Daub4 scaling signal and wavelet) → how different types of signals are optimized for different signal processing tasks

Other Wavelet Functions

< Appendix >

- A set of conditions that a wavelet basis function W must satisfy
- Two of these conditions
  - Related to the time domain

1. Zero average: $\int_{-\infty}^{\infty} W\,dt = 0$

2. Finite energy $\int_{-\infty}^{\infty} |W|^2\,dt < \infty$

   - Usually energy value for both scaling and wavelet signals normalized to 1

     $\rightarrow \int_{-\infty}^{\infty} |W|^2\,dt = 1$

   - Normalized value – allowing to interpret the energy function as a probability density function (PDF) for the process represented by the wavelet

Other Wavelet Functions

< Appendix >

- Two of these conditions
  - Example: the level-1 or level-2 Haar wavelets – clearly satisfy these conditions
    - Averages: $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$ and $\frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 0$, respectively
    - Sum of squares:

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1 \quad \text{and} \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1\text{, respectively}$$

- Third condition
  - Related to the Fourier transform W(ω) of the wavelet

3.Finite norm: $\int_{-\infty}^{\infty} \frac{|W(\omega)|^2}{|\omega|} d\omega < \infty$

Other Wavelet Functions

- Daubechies Wavelet
  - Daub4 scaling signal and wavelet: a support of 4 points
  - Four values Ds4(i) for the scaling signal for Daub4:

$$Ds4(1) = \frac{1+\sqrt{3}}{4\sqrt{2}} \quad Ds4(2) = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

$$Ds4(3) = \frac{3-\sqrt{3}}{4\sqrt{2}} \quad Ds4(4) = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

    - Scaling signals:

$$DS4_1^1 = [Ds4(1), Ds4(2), Ds4(3), Ds4(4), 0, 0, 0, \ldots, 0]$$
$$DS4_2^1 = [0, 0, Ds4(1), Ds4(2), Ds4(3), Ds4(4), 0, \ldots, 0]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$DS4_{(n/2)-1}^1 = [0, 0, 0, \ldots, 0, Ds4(1), Ds4(2), Ds4(3), Ds4(4)]$$
$$DS4_{n/2}^1 = [Ds4(3), Ds4(4), 0, 0, 0, \ldots, 0, Ds4(1), Ds4(2)]$$

      - Level-1 scaling signal – translates in steps of two as does the Haar scaling signal
      - Difference: the coefficients in the last step ($DS4_{n/2}^1$) – wrap around to the beginning of the vector

Other Wavelet Functions

- Daubechies Wavelet
  - Associated Daub4 wavelet:

$$Dw4(1) = \frac{1-\sqrt{3}}{4\sqrt{2}} \quad Dw4(2) = \frac{\sqrt{3}-3}{4\sqrt{2}}$$

$$Dw4(3) = \frac{3+\sqrt{3}}{4\sqrt{2}} \quad Dw4(4) = \frac{-1-\sqrt{3}}{4\sqrt{2}}$$

    - The level-1 translations:

$$DW4_1^1 = [Dw4(1), Dw4(2), Dw4(3), Dw4(4), 0, 0, 0, \ldots, 0]$$

$$DW4_2^1 = [0, 0, Dw4(1), Dw4(2), Dw4(3), Dw4(4), 0, \ldots, 0]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$DW4_{(n/2)-1}^1 = [0, 0, 0, \ldots, 0, Dw4(1), Dw4(2), Dw4(3), Dw4(4)]$$

$$DW4_{n/2}^1 = [Dw4(3), Dw4(4), 0, 0, 0, \ldots, 0, Dw4(1), Dw4(2)]$$

      - The last $DW4_{n/2}^1$ – also warps around

Other Wavelet Functions

- Daubechies Wavelet
  - Example (Fig. 15.5)
    - Results of the level-1 Haar and Daubechies transforms on two types of signal
    - Different talents w.r.t. successfully compressing signals
      - Oscillatory waveform (upper panel) – compressed by Daub4 transform rather well
        - Not much energy left in the fluctuation signal
      - Square wave (lower panel) – more efficiently compressed by the Haar transform

    - Different levels for both the Haar and Daubechies – used to compress signals
      - The more closely the wavelets matches the input curve
        - The closer the difference signal is to zero and
        - The better the quality of the compression in the average signal
      - Judgement for better quality
        - By the level of energy of the original signal preserved by the average or
        - By the (loss of) energy present in the fluctuation

Other Wavelet Functions

- Daubechies Wavelet
  - Example (Fig. 15.5)
    - Progressively higher levels of Daub wavelets – designed to fit higher-order polynomials
      - General rule:
        - Daub*N* wavelet transform – applied to polynomial of the order $< N/2$
        - The input function over the support of a j-level Daub*N* wavelet – a polynomial of the order $< N/2$ → the difference signal (fluctuation) ≈ 0
      - Ex
        - The input signal over the support of the wavelet – largely linear → Daub4 wavelet – usable
        - Quadratic signal → Daub6 wavelet – usable

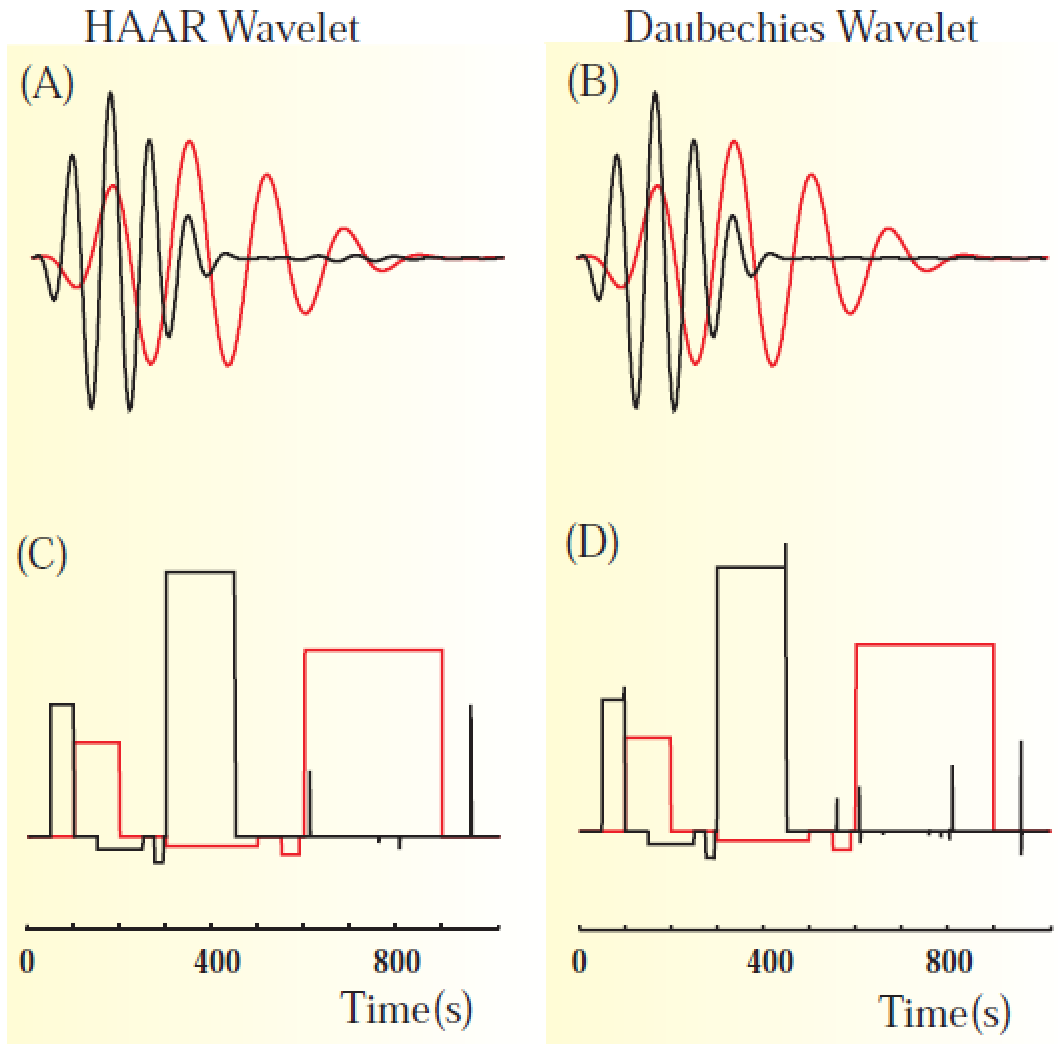

Figure 15.5 Level-1 wavelet transforms of an oscillatory signal and a signal with transients. Both waves were analyzed using the Haar and Daubechies (Daub4) wavelets. The red signal is the original, input, and the black traces represent the fi rst average and difference signals (the same arrangement as in Fig. 15.2). Note that compression of the oscillatory wave in (A) and (B) is done most efficiently by the Daubechies wavelet (i.e., the d1 signal is almost 0 in the latter case). For the wave shown in (C) and (D), the Haar wavelet compresses better. The plots can be obtained with MATLAB scripts haar1.m, haar2.m, daubechies1.m, and daubechies2.m.

Two-dimensional Application

- The same procedure for a one-dimensional WT – applicable to a two-dimensional matrix

1. Image compression
   - Average/trend image – considered a compressed version of the original data
   - Difference/fluctuation image – showing how successfully compression performed

2. Image analysis
   - Difference images – usable as <u>edge detectors</u>
   - Used to <u>enhance edges</u> in images by multiplication of the difference signal of a transformed image with a factor > 1 $\rightarrow$ followed by an <u>inverse transform</u>

Two-dimensional Application

- Example of a WT of an image (Fig. 15.6)
  - Transforming an image with the Haar wavelet
  - The same procedure as $G \overset{H_1}{\mapsto} (a^1 | d^1)$ – used for both the horizontal (rows) and vertical (columns) directions
    - Four new pictures – generated
      - Average (a_H) and fluctuation (f_H) – from the horizontal pass through the data
      - Average (a_V) and fluctuation (f_V) – from the vertical pass
    - Fluctuation in the rows – a tendency to detect vertically oriented transitions (edges)
    - Fluctuations in the columns – detect the horizontally oriented edges
  - Complete two-dimensional procedure
    - Vertical transform on f_H & horizontal transform on f_V → the same image – produced
      - A result where the transform is applied on both the columns and rows)
      - Emphasizing the diagonal fluctuation (f_D)
      - Five images: a_H, a_V, f_H, f_V, and f_D

Two-dimensional Application

- Example of a WT of an image (Fig. 15.6)
  - Complete two-dimensional procedure
    - Sixth image a_1 – obtained from applying the vertical average procedure on a_H or the horizontal procedure on a_V
    - The result of the transform of image I – arranged onto four panels:

$$I \mapsto \left(\frac{a_1 \,|\, f_V}{f_H \,|\, f_D} \right)$$

  - Multiresolution analysis
    - Transform procedure of the original image I – repeated on a_1 → the level-2 transform
      - The upper-left quadrant occupied by a_1 – split again into four subpanels (a_2 and the associated fluctuation signals)
    - Repeating the same transform recursively → MRA applied to images

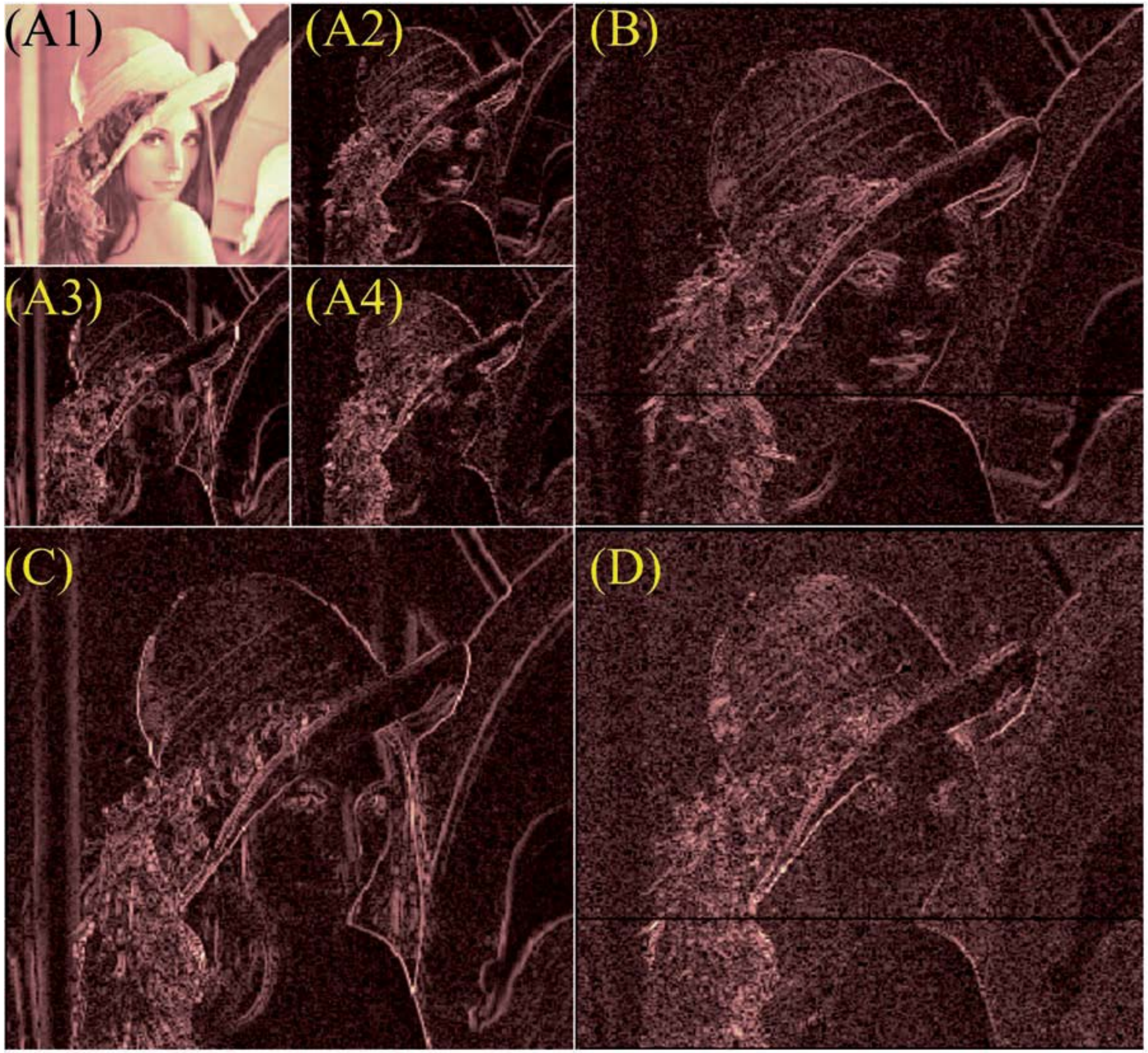

Figure 15.6 Two-dimensional Haar wavelet transform of an image. The top-left panel (A1) shows the trend (compressed) image, and the fluctuations (edges) are shown in the remaining panels. For instance, the top-right panel (B) shows the level-1 fluctuations in the columns (vertical lines) and therefore includes enhanced horizontal edges; the bottom-left panel (C) is the result of level-1 fluctuations along the rows (horizontal lines) and predominantly depicts the vertical edges. The bottom-right panel (D) is a combination of both vertical and horizontal procedures and therefore mainly depicts diagonal edges. The panels (B), (C), and (D) represent the level-1 Haar transform fluctuation, while panels (A2), (A2), and (A3) represent the equivalent fluctuations for the level-2 transform. Accordingly, (A1) depicts the trend result of the level-2 transform.