# Biomedical Signal Processing

## - Wavelet Analysis: Frequency Domain Properties -

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# Introduction

- Digital filter
  - Smoothing a signal by applying a window such as $y(n)=[x(n)+x(n-1)]/2$ in the time domain $\leftrightarrow$ the equivalent in the frequency domain – a low-pass filter

- Wavelet and scaling signals
  - Time domain window of a particular shape – translated over the signal & multiplied with the signal values
  - Spectral composition of scaling and wavelet signals – complementary

# Introduction

- Level-1 Haar scaling signal vs Haar wavelet
  - Haar scaling signal
    - A weighted average of the time domain signal (lower frequency components)
    - Acts as a low-pass filter
  - Haar wavelet
    - A weighted difference signal (higher frequency fluctuations)
    - Band-pass filter – allowing the higher-frequency components in the fluctuation signal
- Comparison of different level wavelets: smaller scale (less dilated) wavelets – higher-frequency components than the large scale ones
- Scaling signal & wavelet correlations with an input signal: complementary
  - Emphasizing the low- and high-frequency components, respectively

# The Continuous Wavelet Transform (CWT)

- Filter bank design
  - Frequency domain properties of wavelets – used
  - Each wavelet at progressively greater scales (more dilated) – passes a narrow, lower band of frequencies from input signal x
- Time domain procedure of the WT
  - To use wavelets w of different scales $\sigma$, move (translate) them over an interval $\tau$ along an input signal, and correlate the wavelet with the input at each of these scales and translations

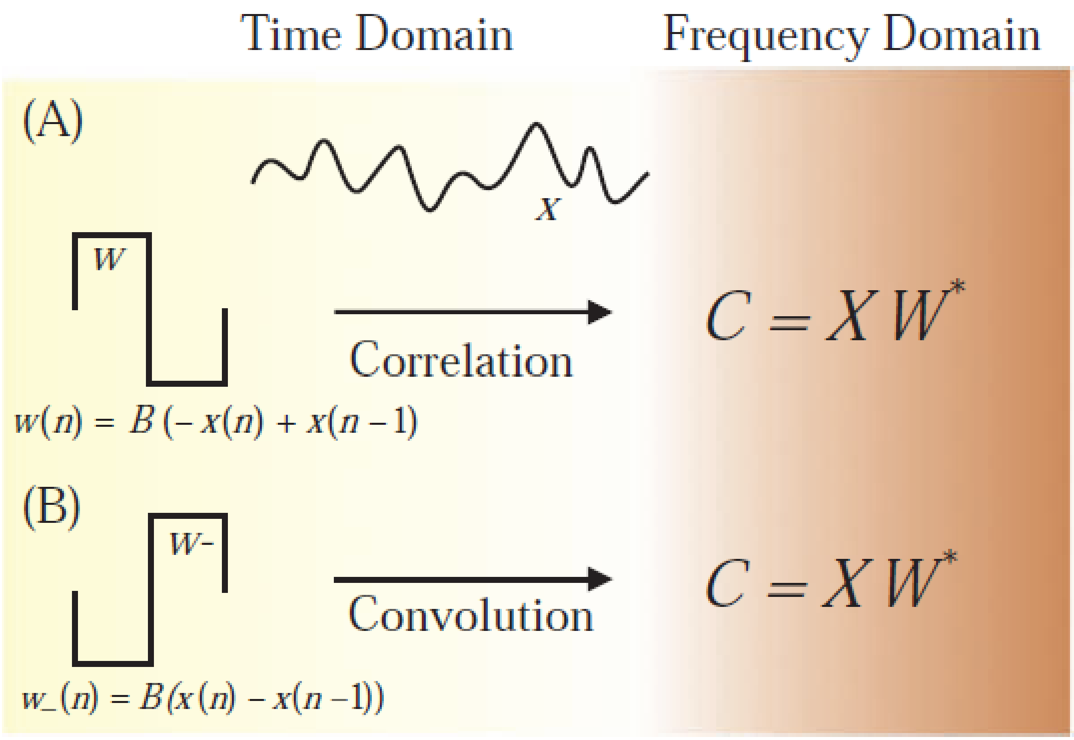



**Figure 16.1** (A) Correlation of a signal *x* with a wavelet w is equivalent to convolution of x with the wavelet's reversed version *w*-, shown in (B). In both cases the frequency domain operation is the product of the Fourier transform of the input *X* with the complex conjugate of the wavelet *W**.

# The Continuous Wavelet Transform (CWT)

- Continuous wavelet transform (CWT)

$$c(\sigma,\tau)=\int_{-\infty}^{\infty} x(t)\frac{1}{\sqrt{\sigma}}w^{*}\left(\frac{t-\tau}{\sigma}\right)dt$$

Complex conjugate of the wavelet
(real wavelet such as Haar and Daubechies wavelet →
* can be omitted)

  - c(σ,τ): the correlation for each τ and σ
    - Using the relationship btw correlation and convolution in the time and frequency domains
  - t-τ – used instead of t+τ (to emphasize translating wavelet from left to right)
  - Scaling term $1/\sqrt{\sigma}$ – used to preserve signal energy across the transform
    - In previous chapter: scale = $2^k$ $\left( W(t)_n^k=\frac{1}{\sqrt{2^k}}W_H\left(\frac{t-2^k n}{2^k}\right) \right)$

# The Continuous Wavelet Transform (CWT)

- Fourier transform pairs:
  - $$c(\sigma,\tau) \Leftrightarrow C; \quad x(t) \Leftrightarrow X; \quad \frac{1}{\sqrt{\sigma}} w\left(\frac{t-\tau}{\sigma}\right) \Leftrightarrow W$$
  - The equivalent of the correlation in the frequency domain: $C = XW^*$

- Reversed wavelet signal in the time domain (w-)
  - Frequency representation – changed:
    - (cosine) even terms: nothing – changed
    - (sine) odd terms: sign – changed
  - Sine terms – the complex terms in the FT → FT of the reversed wavelet = the complex conjugate of the FT of the wavelet:
    - If $$w^\sigma = \frac{1}{\sqrt{\sigma}} w\left(\frac{t-\tau}{\sigma}\right) \Leftrightarrow W$$
    - then $$w_-^\sigma = \frac{1}{\sqrt{\sigma}} w\left(-\frac{t-\tau}{\sigma}\right) \Leftrightarrow W^*$$ ($w^\sigma$ : real ?)

# The Continuous Wavelet Transform (CWT)

- Conclusion

  1. The correlation of x with a wavelet at given scale ($w^{\sigma}$) = the convolution with the reversed wavelet ($w^{\sigma}_{-}$)
     - FT pair for convolution with the reversed wavelet at scale σ:

$$x \otimes w^{\sigma}_{-} \Leftrightarrow XW^{*}$$

  2. Identical frequency domain expression for both the convolution of the input x with a reversed wavelet ($w^{\sigma}_{-}$) and the correlation c of the signal with the (nonreversed) wavelet ($w^{\sigma}$):

$$\begin{tikzcd}
x \otimes w^{\sigma}_{-} \arrow[d, phantom, "|||"] & \Leftrightarrow XW^{*} = C \arrow[dl, bend left] \\
c(\sigma, \tau) &
\end{tikzcd}$$

- Even symmetric wavelets (ex: Mexican Hat wavelet)
  - FT of the wavelet = FT of the reversed wavelet
  - Correlation and convolution of an even wavelet with an input – identical results

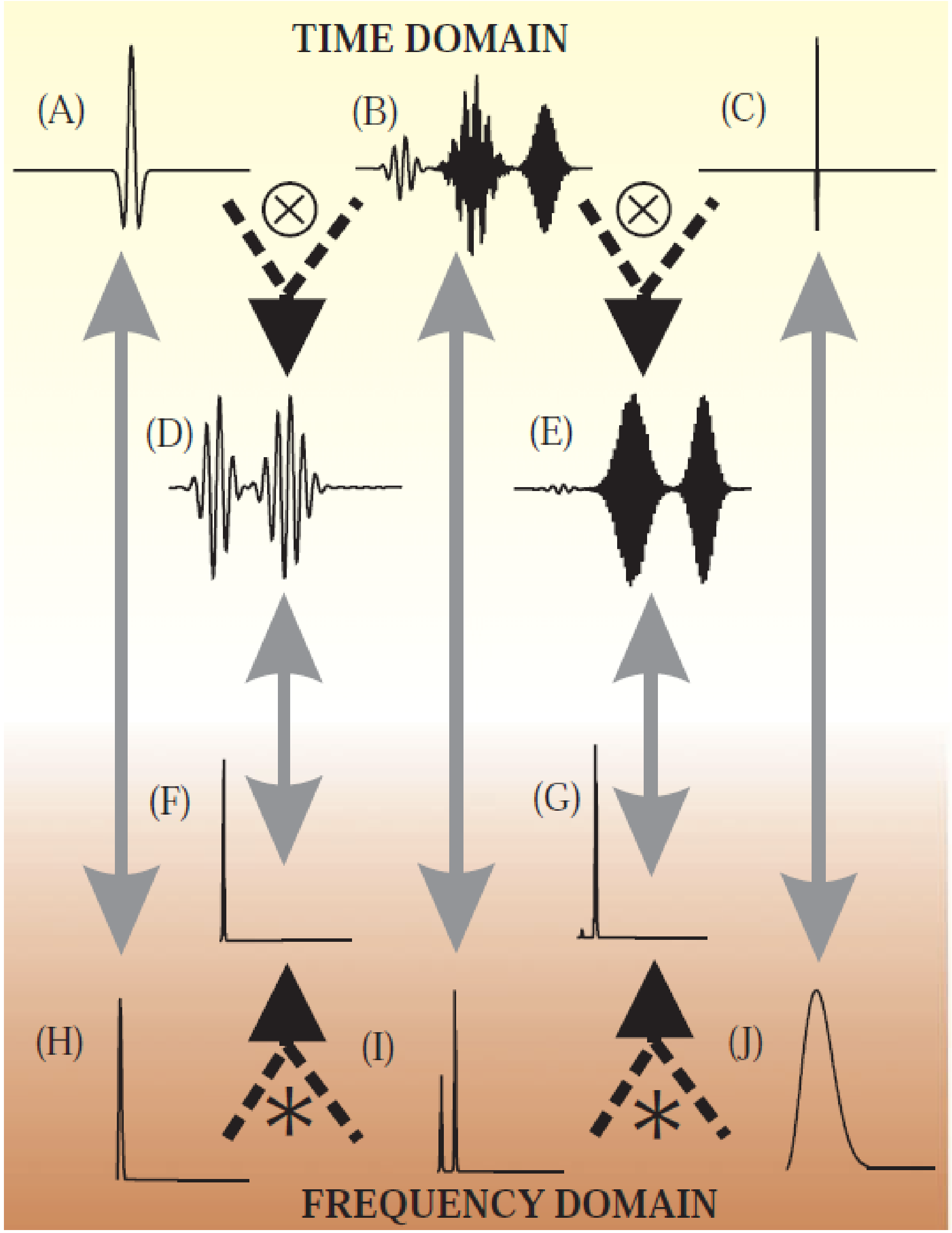
# The Continuous Wavelet Transform (CWT)

- Haar scaling and wavelet-related operations
  - By translating the scaling signal (s) and wavelet (w) over the input (x) signal with one-step increments
    - Previous chapter: jumping in steps of 2 points
  - The output of the scaling signal procedure & wavelet operation:
    - $$s(n) = \frac{1}{\sqrt{2}}(x(n) + x(n-1))$$
    - $$w(n) = \frac{1}{\sqrt{2}}(-x(n) + x(n-1))$$
    - Both procedure: FIR/MA filters
  - Reversing wavelet & scaling signal
    - Wavelet reversal: $w\_(n) = \frac{1}{\sqrt{2}}(x(n) - x(n-1))$
      - May be used for convolution
    - Reversal for scaling signal: $s\_(n) = s(n)$
      - Even symmetric function → no reversal required

# The Continuous Wavelet Transform (CWT)

- Haar scaling and wavelet-related operations
  - Transfer functions in the z-domain
    - Scaling signal: $$H_s(z) = \frac{S(z)}{X(z)} = \frac{1}{\sqrt{2}}(1 + z^{-1})$$
    - Reversed wavelet: $$H_{w-}(z) = \frac{W_-(z)}{X(z)} = \frac{1}{\sqrt{2}}(1 - z^{-1})$$
  - Frequency response
    - Substituting $z = \exp(j\omega T)$ with T (the sample interval)
    - Scaling signals – a low-pass filter characteristic
    - Wavelets – a high-pass filter characteristic
    - Example of two scales of the Mexican Hat wavelet (MHW) (Fig. 16.2)

**Figure 16.2** Example of using wavelet analysis as a filter bank. In this figure we show two scales of the Mexican Hat wavelet (MHW), a higher scale in (A) and a lower one in (C). (The wavelet transform with the example signal (B) shows the transform with higher-scale wavelet and a lower frequency component (D), whereas the lower scale shows the higher frequencies (E)). The preceding operation in the time domain can also be understood from the equivalent operations in the frequency domain. The transform of the original signal in (B) is shown in (I). It can be seen in (I) that there are two frequency components in the original signal. The higher-scale MHW in (A) has a transform containing low frequencies (H), the transform of the other lower scale wavelet in (C) is shifted to the higher frequencies (J). When using the MHW transforms as the filter characteristic, it can be seen that in one case the lower frequencies are predominant (F) and in the other case the higher frequencies predominate (G). The spectra in (F) and (G) correspond with time domain signals (D) and (E), respectively. The gray arrows indicate the Fourier transform pairs. The ⊗ and * symbols represent convolution and multiplication, respectively. Note the following: (1) The vertical scaling is optimized in each panel (i.e., not the same between panels). In the frequency domain, the amplitude spectra (not the raw Fourier transforms) are shown. (2) Because the MHW is an even symmetric function, convolution and correlation with the input are equivalent. (3) There is an inverse relationship between scale and frequency, as compared with the lower-scale wavelet in (C) and frequency plot (J), the higher-scaled wavelet in (A) is associated with the lower-frequency components (H).



# Time Frequency Resolution

- Spectral analysis
  - Spectrum
    - Frequency components in the input signal
    - Whole time domain epoch → uncertain at the exact time location for any particular frequency component
  - Reduction of the size of the epoch of the input signal
    - Increases resolution
    - Necessarily changes the resolution of the spectral analysis

# Time Frequency Resolution

- Illustration for the time-frequency resolution of spectral analysis
  - Sample: a 10-s epoch sampled at 1000 Hz

  - Spectrum with a resolution of 1/10 Hz up to the Nyquist frequency of 500 Hz
  - A spectral peak of a sinusoidal signal with a frequency of 30.06 Hz
    - Indicated by energy in the transform mainly btw 30 Hz and 30.1 Hz
    - We cannot determine where this frequency component occurs in time
      - The 30- to 30.1-Hz component – might be present throughout the 10s epoch) or
      - It could be localized in a burst somewhere within the 10s epoch
  - Conclusion
    - The uncertainty of where the particular signal component occurs in time = 10s
    - The uncertainty about its frequency value = 0.1 Hz (btw 30.0 and 30.1 Hz)

# Time Frequency Resolution

- Illustration for the time-frequency resolution of spectral analysis
  - A reduction of the 10s epoch to a 2s window
    - Time-domain resolution
      - 5× more precise (less uncertain) localization of the spectral components in time
      - Spectral components – located somewhere within a 2s window
    - Frequency-domain resolution
      - ½ Hz up to 500-Hz Nyquist limit
      - The energy of the 30.06 Hz component – in the spectrum mainly btw 30 Hz and 30.5 Hz → increasing the uncertainty about the frequency to 0.5 Hz

# Time Frequency Resolution

- Illustration for the time-frequency resolution of spectral analysis
  - Implication for the Fourier-based spectral analysis
    - The size of the time domain epoch – proportional to the precision with which spectral components can be located in the time domain
      - Time domain resolution of any of the spectral components = the size of the selected epoch
    - The size of the time domain epoch – inversely proportional to the resolution in the frequency domain
      - The spectral resolution = 1/epoch
  - A compromise btw time and frequency resolution
    - For any choice of the epoch length
    - Impossible to choose an epoch length that will accommodate both a high temporal and a high spectral resolution

# Time Frequency Resolution

- Illustration for the time-frequency resolution of spectral analysis
  - A compromise btw time and frequency resolution
    - A very high temporal resolution (small epoch)
      - Always associated with a low spectral resolution & vice versa
    - Choice of long epoch
      - A low frequency – detected (∵ low-frequency components – spread over longer epochs)
      - Higher-frequency components (such as 30.06 Hz)
        - Determined with high precision in the frequency domain
        - Cannot be precisely localized in time
          - The 30.05-Hz component with an intrinsic period only ~33ms – localized within a 10s or 2s windows in the preceding example

# Time Frequency Resolution

- Continuous wavelet transform
  - Improved resolution of high-frequency components in the time domain – compared to a single Fourier transform-based spectral analysis
  - Low-scale wavelet
    - Correlating well with relatively high-frequency components
  - Higher-scale wavelet (more stretched scale)
    - Correlating better with the lower-frequency components

# Time Frequency Resolution

- Continuous wavelet transform
  - Wavelet scale ($\sigma$)
    - The frequency associated with a particular wavelet scale $\propto$the inverse of the scale
    - Often expressed as $2^k$
      - Every subsequent integer value of k → a factor 2 difference in the frequency (an octave difference in musical terms)
      - All non-integer values btw k and k+1 – steps (voices) within the octave
        - In some applications: the scale – indicated as $2^{n/\nu}$ with $n=1,2,3,\ldots,n\times\nu$ ($\nu$ – number of voices)

# Time Frequency Resolution

- Illustration of the uncertainty principle (Fig. 16.2)
  - Low- and high-frequency components – distributed in different periods of the input signal epoch (Fig. 16.2B)
  - Power spectrum of the entire epoch (Fig. 16.2I)
    - The presence of spectral components
    - Without indicating where in the epoch they occur
  - MHW transform (Fig. 16.2D & E)
    - More precisely indicating (with less uncertainty) where each component is located in time
- Empirically measured time series
  - Spectral components – unknown a priori → we do not know in advance what scale(s) of wavelet to select
  - Solution: to explore a range of scales, similar to the FT or a filter bank where sets of frequencies are considered

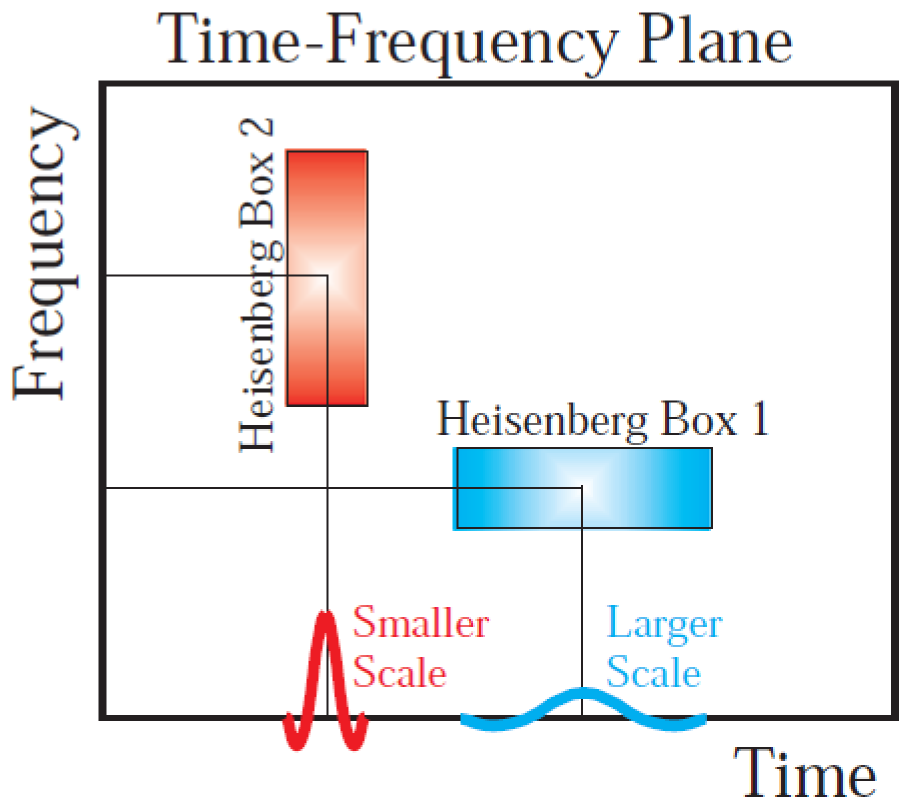



**Figure 16.3** The time frequency plane and Heisenberg uncertainty boxes for a low frequency signal component poorly localized in time but with reasonable spectral resolution (box 1) and for a higher-frequency tone with better time resolution but lower spectral resolution (box 2). This figure also demonstrates the features of the scalogram (Section 16.4) in which low-frequency components (longer period) are detected by larger scale wavelets (blue) and higher-frequency components are detected by smaller scale wavelets (red). This procedure creates a time resolution, which is appropriate for each frequency — that is, long epochs for slow oscillations and shorter ones for faster oscillations. In contrast, the classical Fourier transform-based spectrum has a fixed Heisenberg box for all frequencies determined by its epoch length (see the examples in Section 16.3).

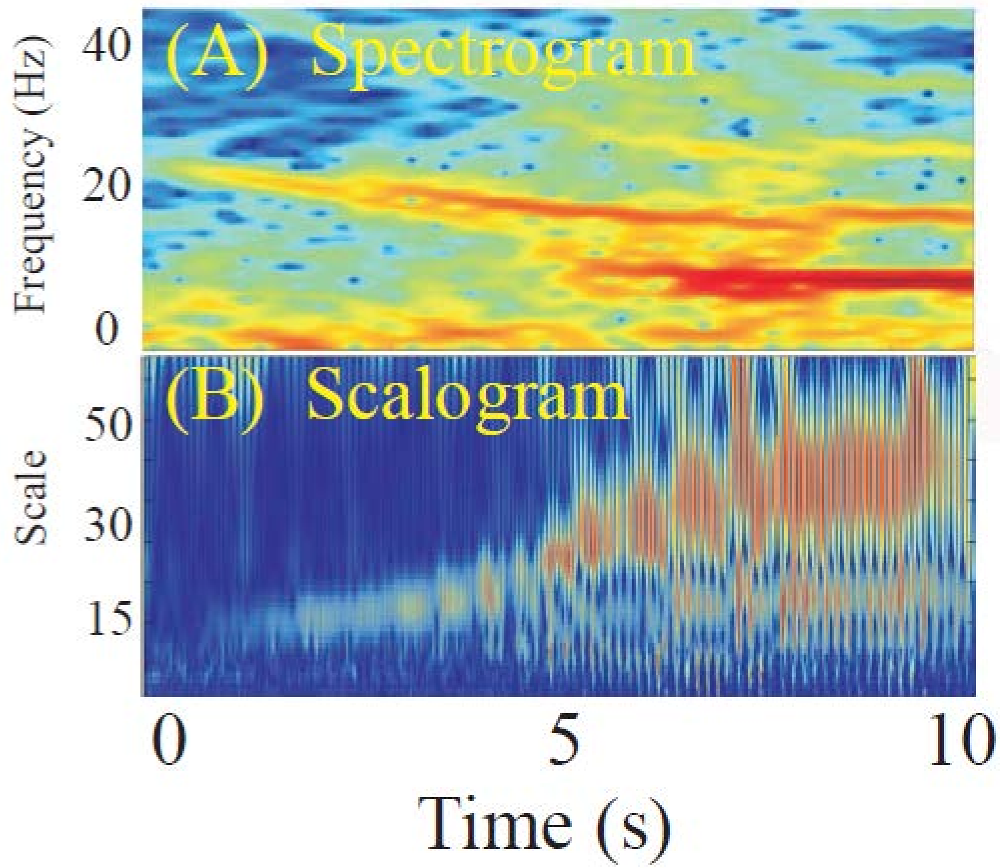



**Figure 16.5** A spectrogram (A) and scalogram (B) of an EEG signal during the onset of an epileptic seizure.

The End of Class MD5301

(Thank You !)